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ON A PROBLEM OF GILLMAN AND KEISLER

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We shall obtain a result concerning the question 3B of Keisler [6].

If S is a set, then $|S|$ is the cardinality of S. The axiom of choice is assumed throughout. For the notion of an ultrafilter over a set S we refer e.g. to Keisler [6], p. 115. Suppose now that $|S| = m$ is infinite and $\mathcal{U}$ is an ultrafilter over S. $\mathcal{U}$ is said to be uniform if for every $X \in \mathcal{U}$, $|X| = m$. Keisler [6] calls an ultrafilter $\mathcal{U}$ over S regular if there is a set $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| = m$ and the intersection of any infinitely many sets from $\mathcal{V}$ is empty.

Now we can state the question 3B, [6]. Is it true that for any infinite S, every uniform ultrafilter over S is regular? Keisler [6] states that the answer is known only in the case when $|S| = \aleph_0$. The answer is then trivially positive. In the present paper we shall consider the next case, namely $|S| = \aleph_1$. Our main result is then as follows. If Gödel's axiom of constructibility, V = L, is true, then every uniform ultrafilter over S is regular. Using the methods of Cohen [2], models of set theory can be constructed in which "V = L" is false and "if $|S| = \aleph_1$, then every uniform ultrafilter over S is regular" is true.

The question, "if $|S| = \aleph_1$, is every uniform ultrafilter over S regular?", also appears in Erdös, Hajnal [3] as Problem 82. The wording there is different. The question is attributed to L.Gillman. No reference is given. Thus we answer Gillman's question under the assumption that V = L.

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An ordinal is the set of smaller ordinals. Small Greek letters will denote ordinals. Sometimes, for the sake of greater explicitness, we write $\{\eta | \eta < \alpha\}$ instead of just α. ω_0 denotes the least infinite ordinal, ω_1 the least uncountable ordinal. If f is a function, then $\mathcal{D}(f)$ is the domain of f. If $X \subseteq \mathcal{D}(f)$, then $f | X$ is f restricted to X.

The paper has two sections. In the Section 1 we show that the positive answer to the above problem follows from a certain combinatorial hypothesis. This hypothesis is related to the so-called Kurepa's hypothesis: There is a family $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$ such that $|\mathcal{F}| = \aleph_2$ and $|\{X \cap \alpha \mid X \in \mathcal{F}\}| = \aleph_0$ for every $\alpha \in \omega_1$. The main part of the proof is the construction of a matrix of sets with the properties similar to those discussed in Chang [1], Erdös, Ulam [4] or Ulam [9].

In the Section 2 we prove that the combinatorial hypothesis mentioned above follows from V = L. This is done by a slight modification of R.Solovay's proof that V = L implies Kurepa's hypothesis, [8].

Section 1

Lemma 1. *For every $\alpha \in \omega_1$ let F_α be a family of functions satisfying the following conditions.*

(a) $|F_\alpha| = \aleph_0$,

(b) $\forall \varphi \in F_\alpha (\varphi \subseteq \alpha \times \alpha$,

Then there is a family

$$\{A(\alpha, \beta) : \alpha \in \omega_1, \beta \in \omega_1\}$$

of subsets of ω_1 with the following properties:

(A) *For every $\alpha \in \omega_1$ and every infinite subset S of $\omega_1 - \alpha$, $\bigcap \{A(\alpha, \beta) : \beta \in S\}$ is countable. Moreover, if θ is the least ordinal such that $\theta \cap S$ is infinite then $\bigcap \{A(\alpha, \beta) : \beta \in S\} \subseteq \theta$.*

(B) *If f is a function from ω_1 to ω_1 such that for each α there is a $\varphi \in F_\alpha$ with $f \cap (\alpha \times \alpha) \subseteq \varphi$, then for each ifninite set $S \subset \mathcal{D}(f)$,*

$$\bigcup \{A(\alpha, f(\alpha)) : \alpha \in S\} = \omega_1 .$$

Proof of Lemma 1. We may enumerate F_α in the form $\{\varphi(\alpha, k) : k \in \omega\}$. Let us also fix an enumeration $\{\alpha_n : n \in \omega\}$ of each ordinal $\alpha \in \omega_1$. We define

$$A(\alpha, \beta) = \{\gamma : (\gamma \leq \alpha) \quad \text{or} \quad (\gamma \leq \beta) \quad \text{or} \quad (\alpha, \beta < \gamma; \alpha = \gamma_n ; \text{ and } \exists m < n(\varphi(\gamma, m)(\alpha) = \beta))\}$$

To prove (A) we suppose that S is an infinite subset of $\omega_1 - \alpha$ and that θ is the least ordinal such that $S \cap \theta$ is infinite. If (A) were false then there would be a γ such that $\theta \leq \gamma$ and $\gamma \in \bigcap \{A(\alpha, \beta) : \beta \in S\}$. For some n, α is γ_n. Thus, since γ is in this intersection, it must be that for each $\beta \in S \cap \gamma$ there is an m less than n for which $\varphi(\gamma, m)(\alpha) = \beta$. This however is impossible; for there are only finitely many such m's which must serve for infinitely many such β's.

To prove (B), let f and S satisfy the premise of (B). Let $\eta = \bigcup S \cup \bigcup \{f(s) : s \in S\}$. It is obvious that $\eta \subseteq \bigcup \{A(\alpha, f(\alpha)) : \alpha \in S\}$; we must still show that

$$\omega_1 - \eta \subseteq \bigcup \{A(\alpha, f(\alpha)) : \alpha \in S\}.$$

Suppose that $\eta \leq \gamma < \omega_1$. There is some n such that $f \cap (\gamma \times \gamma) = \varphi(\gamma, n)$. Because $S \cap \gamma = S$, we may write S in the form $\{\gamma_\nu : \nu \in N\}$. Select any ρ with $n < \rho \in N$. Surely

$$f(\gamma_\rho) = \varphi(\gamma, n)(\gamma_\rho).$$

But this means that

$$\gamma \in \bigcup \{A(\alpha, f(\alpha)) : \alpha \in S\}.$$

Lemma 2. *Suppose that there is a sequence $\{\mathcal{F}_\alpha\}$ satisfying the conditions of Lemma 1 and the following additional condition:* (C) *for every $g \in \omega_1{}^{\omega_1}$ there is an f satisfying the premise of* (B) *and such that for every $\gamma \in \mathcal{D}(f)$, $g(\gamma) < f(\gamma)$. Then every uniform ultrafilter over a set of cardinality $\aleph_1$ is regular.*

Proof. For the application of Lemma 1, choose the sequence $\{\mathcal{F}_\alpha\}$ as in Lemma 2. Then let

$$\{A(\gamma,\xi) \mid \gamma \in \omega_1, \xi \in \omega_1\}$$

be a family of sets guaranteed by Lemma 1.

Suppose that $\mathcal{U}$ is a uniform ultrafilter on ω_1; suppose too that we have defined sets $A(\alpha,\beta)$ as in Lemma 1.

In case there is an α such that $X = \{\beta : A(\alpha,\beta) \in \mathcal{U}\}$ is uncountable, then the set $\{A(\alpha,\beta) - \beta : \beta \in X - \alpha\}$ renders $\mathcal{U}$ regular. To see this let S be a countable subset of X and let θ be the first ordinal such that $S \cap \theta$ is infinite. We have seen that

$$\bigcap\{A(\alpha,\beta) - \beta : \beta \in S\} \subseteq \bigcap\{A(\alpha,\beta) : \beta \in S\} \subseteq \theta\ .$$

In addition, if $\xi \in \theta$, there is a $\beta \in S$ with $\xi < \beta < \theta$. Thus, of course, $\xi \notin A(\alpha,\beta) - \beta$; surely $\xi \notin \bigcap\{A(\alpha,\beta) - \beta : \beta \in S\}$.

If there is no such ordinal α, there is a different collection which renders $\mathcal{U}$ regular. In this case the function $g(\alpha) = \cup\{\beta : A(\alpha,\beta) \in \mathcal{U}\}$ is a well-defined function. By the assumption (C) there is a function f satisfying the premise of (B) for which $g(\alpha) < f(\alpha)$ for each α.

In this case $\{\omega_1 - A(\alpha, f(\alpha)) : \alpha \in \omega_1\}$ renders $\mathcal{U}$ regular. This fact follows from (B) since for every infinite S

$$\begin{aligned}\bigcap\{\omega_1 - A(\alpha,f(\alpha)) : \alpha \in S\} &= \omega_1 - \bigcup\{A(\alpha,f(\alpha)) : \alpha \in S\}\\ &= \omega_1 - \omega_1 = 0\ .\end{aligned}$$

This completes Lemma 2.

Let (P) be the following property of a sequence $\{F_\alpha\}$: $\{F_\alpha\}$ satisfies the conditions stated in Lemma 1 and, in addition, for each function $g \in \omega_1{}^{\omega_1}$ there is a function f such that

$$(\forall\alpha \in \omega_1)(\exists\varphi \in F_\alpha)(f \cap (\alpha \times \alpha) \subseteq \varphi)\ ;$$

and $\{\gamma : \gamma \in \mathcal{D}(f)\ \&\ g(\gamma) < f(\gamma)\}$ is uncountable. In the manner above one can show that every uniform ultrafilter on ω_1 is regular provided that there is a family having the property (P).

Section 2

Our goal is to prove the following.

Theorem 2. *If* $V = L$, *then there is a set* $\mathcal{F} \subseteq \omega_1{}^{\omega_1}$ *such that*

$$(\forall \alpha \in \omega_1)\,[\,|\,[f \cap (\alpha \times \alpha) \mid f \in \mathcal{F}\,]\,| \leq \aleph_0\,]\ ;$$

$$(\forall g \in \omega_1{}^{\omega_1})(\exists f \in \mathcal{F})(\forall \alpha \in \omega_1)[g(\alpha) < f(\alpha)]\ .$$

Definition 1. (R.Solovay, [8]). For each $\xi \in \omega_1$ let M_ξ be the smallest initial segment of L_{ω_1} such that $\langle M_\xi, \epsilon\rangle \prec \langle L_{\omega_1}, \epsilon\rangle$ and $\xi \in M_\xi$. L_α denotes the family of sets constructible before stage α. See e.g. [5].

The next definition is motivated by [8].

Definition 2. Let

$$\mathcal{F} = \{f \mid f \in \omega_1{}^{\omega_1}\ \&\ (\forall \xi \in \omega_1)\,[f \cap (\xi \times \xi) \in M_\xi\,]\}\ .$$

It is known that there is a formula $\varphi(x)$ of set theory, with one free variable x, such that

$$f \in \mathcal{F} \iff \langle L_{\omega_2}, \epsilon\rangle \models \varphi(f)\ .$$

Definition 3. [8]. $\langle N_0, \epsilon\rangle$ is the minimal elementary submodel of $\langle L_{\omega_2}, \epsilon\rangle$. Suppose that $\langle N_\rho, \epsilon\rangle$ has been defined and $|N_\rho| = \aleph_0$. Let λ_ρ be the least ordinal such that $\lambda_\rho \notin N_\rho$. Then $N_{\rho+1}$ is the Skolem hull in L_{ω_2} of $N_\rho \cup \{\lambda_\rho\}$. If ρ is a limit ordinal, then

$$N_\rho = \bigcup_{\sigma \in \rho} N_\sigma\ .$$

Hence for each $\rho \in \omega_1$,

$$\langle N_\rho, \epsilon \rangle \prec \langle L_{\omega_2}, \epsilon \rangle . \qquad (*)$$

Let

$$K = \{ \lambda_\rho \mid \rho \in \omega_1 \} \cdot$$

Lemma 1. [8]. K *is a closed unbounded subset of* ω_1.

Proof is left to the reader.

Definition 4. [8]. For each $\rho \in \omega_1$, $\langle N'_\rho, \epsilon \rangle$ is the transitive realization of $\langle N_\rho, \epsilon \rangle$. Thus

$$\langle N'_\rho, \epsilon \rangle \cong \langle N_\rho, \epsilon \rangle .$$

Lemma 2. [8]. $N_\rho \cap \omega_1 = \lambda_\rho = \omega_1^{N'_\rho}$.

Proof. The second equality is an immediate consequence of the first one and of Definition 4. To see that $N_\rho \cap \omega_1 = \lambda_\rho$, suppose that $\alpha \in N_\rho$ and $\beta \in \alpha$. It follows from Definition 3, (*), that the first h (first with respect to Gödel's ordering of L) such that

$$h : \omega_0 \xrightarrow[1-1]{\text{onto}} \alpha$$

belongs to N_ρ. Again by (*), $h(n) \in N_\rho$ for every $n \in \omega_0$. Thus $\alpha \subseteq N_\rho$. The rest follows easily.

The next lemma is the main part of the whole proof of Theorem 2.

Lemma 3. [8]. *For each* $\xi \in \omega_1$, $K \cap \xi \in M_\xi$.

Proof. *Part* I. Let $\lambda = \sup (K \cap \xi)$. It follows from Lemma 1 that $\lambda \in K$. Thus for some $\eta \in \omega_1$, $\lambda = \lambda_\eta$. We shall prove that $N'_\eta \in M_\xi$. We have $\langle N'_\eta, \epsilon \rangle \equiv \langle L_{\omega_2}, \epsilon \rangle$ and $\lambda_\eta = \omega_1^{N'_\eta}$ (see Definition 3, Definition 4 and Lemma 2). Hence for every $a \subseteq \omega_0$, if $a \in N'_\rho$, then a is constructed before stage λ_η. However, because $\langle M_\xi, \epsilon \rangle \prec \langle L_{\omega_1}, \epsilon \rangle$, new sets of natural numbers are constructed cofinally, in M_ξ. It follows that N'_η is a proper initial segment of M_ξ and thus $N'_\eta \in M_\xi$.

Part II. Similarly as in Definition 3, we can define an elementary chain $\langle P_\rho, \epsilon\rangle$, $\rho < \eta$, such that $\langle P_\rho, \epsilon\rangle \prec \langle N'_\eta, \epsilon\rangle$. Since $N'_\eta \in M_\xi$, this can be done in M_ξ. Then in M_ξ we can also define the transitive realization, P'_ρ, of P_ρ. $\rho < \eta$. It can be shown by induction that for each $\rho < \eta$,

$$(N_\eta, N_\rho) \cong (N'_\eta, P_\rho).$$

This is so because $N_{\rho+1}$ is the Skolem hull of $N_\rho \cup \{\lambda_\rho\}$ in $N_\eta \prec L_{\omega_2}$ and $P_{\rho+1}$ is the Skolem hull of $P_\rho \cup \{\text{1st ordinal} \notin P_\rho\}$ in N'_η. Therefore, for all $\rho < \eta$, $P'_\rho = N'_\rho$. We have thus shown that $\langle N'_\rho : \rho < \eta\rangle \in M_\xi$. Hence also

$$\langle \omega_1^{N'_\rho} : \rho < \eta\rangle = \langle \lambda_\rho : \rho < \eta\rangle \in M_\xi .$$

Since either

$$K \cap \xi = \{\lambda_\rho : \rho < \eta\} ,$$

or

$$K \cap \xi = \{\lambda_\rho : \rho \leq \eta\} ,$$

$K \cap \xi \in M_\xi$ as well.

This completes the proof of Lemma 3.

Definition 5. We define a function $h \in \omega_1{}^{\omega_1}$ as follows:

$$h(\eta) = \lambda_0 \quad \text{for} \quad \eta < \lambda_0 ;$$

$$h(\eta) = \lambda_{\rho+1} \quad \text{for} \quad \lambda_\rho \leq \eta < \lambda_{\rho+1} .$$

Lemma 4. *For each* $\xi \in \omega_1$, $h \cap (\xi \times \xi) \in M_\xi$.

Proof. As in Lemma 3, $\lambda_\eta = \sup (K \cap \xi)$. It follows easily from Lemma 3 that $h | \lambda_\eta \in M_\xi$. We claim that $h \cap (\xi \times \xi) = h | \lambda_\eta$. This follows from the fact that for $\zeta \geq \lambda_\eta$, $h(\zeta) \geq \lambda_{\eta+1} > \xi$. The conclusion follows.

Lemma 5. (i) $h \in \mathcal{F}$.
(ii) $\langle L_{\omega_2}, \epsilon \rangle \models \varphi(h)$.

Proof. Lemma 4 and Definition 2.

Proof of Theorem 2. It is sufficient to take $\mathcal{F}$ as in Definition 2. Assume the contrary. Then the first $g \in \omega_1{}^{\omega_1}$ (compare the proof of Lemma 2) such that

$$\langle L_{\omega_2}, \epsilon \rangle \models (\forall f)[\varphi(f) \rightarrow (\exists \eta \in \omega_1)(g(\eta) \geq f(\eta))] \qquad (**)$$

is a definable element of L_{ω_2}. Thus by (*), $g \in N_\rho$ for each $\rho \in \omega_1$. Applying (*) once again, if $\eta \in \omega_1 \cap N_\rho$, then $g(\eta) \in \omega_1 \cap N_\rho$. But by Lemma 2, $\omega_1 \cap N_\rho = \lambda_\rho$. Hence we have shown that for all η, if $\eta < \lambda_\rho$, then $g(\eta) < \lambda_\rho$. This holds for all $\rho \in \omega_1$. It is now easy to see that

$$(\forall \eta \in \omega_1)(g(\eta) < h(\eta)) .$$

Combining this with Lemma 5, (ii), we get

$$\langle L_{\omega_2}, \epsilon \rangle \models (\exists f)[\varphi(f) \;\&\; (\forall \eta \in \omega_1)(g(\eta) < f(\eta))] .$$

This contradicts (**) and Theorem 2 is thus proved.

The next lemma is easy to prove.

Lemma 6. *Let $\mathcal{F}$ be as in* Theorem 2 *and for every $\alpha \in \omega_1$, set*

$$\mathcal{F}_\alpha = \{f \cap (\alpha \times \alpha) \mid f \in \mathcal{F}\} .$$

Then $\{\mathcal{F}_\alpha\}$ satisfies (P) (see Section 1).

Theorem 3. *If* V = L, *then every uniform ultrafilter over a set of cardinality $\aleph_1$ is regular.*

Proof. This follows from Theorem 1, Lemma 6 and Theorem 2.

Other results concerning the regularity of ultrafilters are described in [7], Chapter I, § 6.

References

[1] C.C.Chang, Descendingly incomplete ultrafilters, Trans. of the Am. Math. Soc. 126 (1967) 108–118.
[2] P.J.Cohen, The independence of the continuum hypothesis I, II. Proc. Nat. Acad. Sci., USA 50 (1963) 1143–1148; 51 (1964) 105–110.
[3] P.Erdös and A.Hajnal, Unsolved problems in set theory, Proc. of the 1967 Summer Inst. in UCLA, ed. D.Scott, to appear.
[4] P.Erdös and S.Ulam, On equations with sets as unknowns, Proc. Nat. Acad. Sci. USA 60 (1968) 1189–1195.
[5] K.Gödel, The Consistency of the Generalized Continuum Hypothesis with the Axioms of Set Theory (Princeton, 1958).
[6] H.J.Keisler, A survey of ultraproducts, Logic, Methodology and Philosophy of Science, Proc. of the 1964 Int. Congress, ed. Y.Bar-Hillel (North-Holland, Amsterdam, 1965) pp. 112–126.
[7] K.Prikry, Changing measurable into accessible cardinals, Doctoral dissertation, Univ. of California, 1968. To appear.
[8] R.Solovay, Notes by H.J.Keisler from a lecture delivered in Madison, 1969.
[9] S.Ulam, Zur Masstheorie in der Algemeinen Mengenlehre, Fund. Math. 16 (1930) 140–150.